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On the L^p Space of Observables on Product MV Algebras[†]

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A weakly σ-distributive product MV algebra M is considered as a base of a quantum structure model. A state is a morphism from M to the unit interval, and an observable is a morphism from the system of all Borel sets to M. It is proved that the subspace L^p of the space of observables is a complete pseudometric space. This result generalizes the previous result; the proof is new.

1. INTRODUCTION

There is given an MV algebra $(M, \oplus, \odot, *, 0, 1)$, where $\oplus$ and $\odot$ are binary operations, $*$ is a unary operation, and 0 and 1 are fixed elements such that some axioms are satisfied [1, 2, 7]. By the Mundici representation theorem [4] there exists a commutative l-group G such that $M \simeq \langle 0, u \rangle \subset G$, where 0 is the neutral element of G, and u is a strong unit in G,

$$a \oplus b = (a + b) \wedge u$$

$$a \odot b = (a + b - u) \vee 0$$

$$a^* = u - a$$

An MV algebra M is called a product MV algebra [6] if there is given a binary operation $\cdot$ on M satisfying the following conditions:

(i) $u \cdot u = u$.
(ii) The operation $\cdot$ is commutative and associative.
(iii) If $a + b \leq u$, then $c \cdot (a + b) = c \cdot a + c \cdot b$ for any $x \in M$.

[†] This paper is dedicated to the memory of Prof. G. T. Rüttimann.

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(iv) If $a_n \searrow 0$, $b_n \searrow 0$, then $a_n \cdot b_n \searrow 0$.

A state is a mapping $m: M \to \langle 0, 1 \rangle$ satisfying the following conditions:

(i) $m(u) = 1$
(ii) If $a = b + c$, then $m(a) = m(b) + m(c)$.
(iii) If $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

An observable is a mapping $x: \mathcal{B}(R) \to M$ ($\mathcal{B}(R)$ is the σ-algebra of Borel subsets of R) such that the following properties are satisfied:

(i) $x\,(R) = u$.
(ii) If $A \cap B = \emptyset$, then $x\,(A \cup B) = x(A) + (B)$.
(iii) If $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

If $m: M \to \langle 0, 1 \rangle$ is a state and $x: \mathcal{B}(R) \to M$ is an observable, then $m_x = m \circ x: \mathcal{B}(R) \to \langle 0, 1 \rangle$ is a probability measure.

An observable $x: \mathcal{B}(R) \to M$ belongs to L^p if there exists

$$\int_R |t|^p \, dm_x(t)$$

The notion of an observable is a generalization of the notion of a random variable $\xi: (\Omega, \mathcal{S}, P) \to (R, \mathcal{B}(R), P_\xi)$, since

$$\int_\Omega |\xi|^p \, dP = \int_R |t|^p \, dP_\xi(t)$$

In the L^p space of random variables the distance is defined by the formula

$$\rho_p(\xi, \eta)^p = \int_\Omega |\xi - \eta|^p \, dP = \iint_{R^2} |u - v|^p \, dP_T(u, v)$$

where $T = (\xi, \eta): \Omega \to R^2$, $P_T: \mathcal{B}(R^2) \to \langle 0, 1 \rangle$, $P_T(A) = P(T^{-1}(A))$. Instead of a random variable ξ, we consider an observable, and instead of a random vector T we consider a so-called joint observable. The joint observable of observables $x, y: \mathcal{B}(R) \to M$ is a mapping $h: \mathcal{B}(R^2) \to M$ satisfying the following conditions:

(i) $h(R^2) = u$.
(ii) If $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$.
(iii) If $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$.
(iv) $h(C \times D) = x(C) \cdot y\,(D)$ for any $C, D \in \mathcal{B}(R)$.

Now it is natural to define the distance of two observables $x, y \in L^p$ by the formula

$$\rho(x, y) = \begin{cases} \left(\int\int_{R^2} |u - v|^p \, dm \circ h(u, v)\right)^{1/p} & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

Here h is the joint observable of x, y. Of course, we have to prove that the joint observable exists and the function $(u, v) \mapsto |u - v|^p$ is integrable with respect to $m \circ h$. It works in so-called weakly σ-distributive MV algebras. A σ-complete MV algebra is weakly σ-distributive if for any bounded double sequence (a_{ij}) of elements of M such that $a_{ij} \downarrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) we have

$$\bigwedge_{\varphi: N \to N} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0$$

Lemma 1. Let M be weakly σ-distributive product MV algebra. Then to every observables x, y: $\mathcal{B}(R) \to M$ there exists their joint observable.

Proof. [6]. ■

Lemma 2. If x, y are observables from L^p, then g: $(u, v) \mapsto |u - v|^p$ is integrable with respect to $m \circ h$.

Proof. Consider the probability space $(R^2, \mathcal{B}(R^2), m \circ h)$ and random variables ξ, η defined on the space by the formulas

$$\xi(u, v) = u, \qquad \eta(u, v) = v$$

Evidently

$$\begin{aligned} P_\xi(A) &= P(\xi^{-1}(A)) = m(h\,(A \times R)) \\ &= m(x(A)y(R)) = m_x(A) \end{aligned}$$

hence

$$\int_{R^2} |\xi|^p \, dP = \int_R |t|^p \, dP_\xi(t) = \int_R |t|^p \, dm_x(t)$$

We have obtained that $\xi \in L^p$. Similarly $\eta \in L^P$. Therefore $\xi - \eta \in L^p$, hence

$$\int\int_{R^2} |u - v|^p \, dm \circ h(u, v) = \int\int_{R^2} |\xi - \eta|^p \, dP < \infty \quad ■$$

Recall that in ref. 5 the distance $\rho(x, y)$ was defined by the formula

$$\rho(x, y) = \left(\int_R |t|^p \, dm_{x-y}(t)\right)^{1/p}$$

where $x - y$ is the difference of observables. It can be defined by the formula

$$x - y = h \circ g^{-1}$$

where h is the joint observable of the observables x, y and g: $R^2 \to R$ is defined by $g(u, v) = u - v$. Of course,

$$\int_R |t|^p \, dm_{x-y}(t) = \int_R |t|^p \, dm \circ h \circ g^{-1}\,(t)$$

$$= \iint_{R^2} |g|^p \, dm \circ h = \iint_{R^2} |u - v|^p \, dm \circ h(u, v)$$

2. COMPLETENESS OF L^p

Theorem. Let M be a weakly σ-distributive product MV algebra. Then (L^p, ρ) is a complete pseudometric space.

Proof. To prove the symmetry, take φ: $R^2 \to R^2$, $\varphi(u, v) = (v, u)$. Then

$$h \circ \varphi^{-1}\,(A \times B) = h(B \times A) = x(B)y(A)$$

hence $h_1 = h \circ \varphi^{-1}$ is the joint observable of observables y, x. If we put $g(u, v) = |u - v|^p$, then

$$\rho(x, y)^p = \int\int_{R^2} g \, dm \circ h = \int\int_{R^2} g \circ \varphi \, dm \circ h \circ \varphi^{-1}$$

$$= \int\int_{R^2} g \, dm \circ h_1 = \rho(y, x)^p$$

To prove the triangle inequality, let us mention first

$$\rho(x, y)^p = \int\int_{R^2} |u - v|^p \, dm \circ h_{xy}(u, v) = \int\int\int_{R^3} |u - v|^p \, dm \circ h(u, v, w)$$

where h: $\mathcal{B}(R^3) \to M$ is such a morphism that $h(A \times B \times C) = x(A)y(B)z(C)$. Similarly,

$$\rho(x, z)^p = \int\int\int_{R^3} |u - w|^p \, dm \circ h(u, v, w)$$

$$\rho(y, z)^p = \int\int\int_{R^3} |v - w|^p \, dm \circ h(u, v, w)$$

Consider $(R^3, \mathcal{B}(R^3), m \circ h)$ and put $\xi(u, v, w) = u$, $\eta(u, v, w) = v$, $\zeta(u, v, w) = w$. We obtain

$$\rho(x, y)^p = \int_{R^3} |\xi - \eta|^p \, dP$$

$$\rho(x, z)^p = \int_{R^3} |\xi - \zeta|^p \, dP$$

$$\rho(u, x)^p = \int_{R^3} |\eta - \zeta|^p \, dP$$

Using the triangle inequality in the space $L^p(R^3, \mathcal{B}(R^3)P)$, we obtain

$$\rho(x, y) = \left(\int_{R^3} |\xi - \eta|^p \, dP\right)^{1/p}$$
$$\leq \left(\int_{R^3} |\xi - \zeta|^p \, dP\right)^{1/p} + \left(\int_{R^3} |\zeta - \eta|^p \, dP\right)^{1/p} = \rho(x, z) + \rho(z, y)$$

Now, let $(x_n)_n$ be a Cauchy sequence in L^p, i.e. $\lim_{n,k\to\infty} \rho(x_n, x_k) = 0$. We shall work with the space $(R^N, \sigma(\mathcal{C}), P)$, where $\sigma(\mathcal{C})$ is the σ-algebra generated by cylinders and P is the measure induced by the consistent system of measures

$$P_n = m \circ h_n\colon \quad \mathcal{B}(R^n) \to \langle 0, 1\rangle$$

where h_n is the joint observable of $x_1, \ldots, x_n$, hence

$$P_n(A_1 \times \cdots \times A_n) = m(x_1(A_1) \cdot \cdots \cdot x_n(A_n))$$

Define further $\xi_n\colon R^N \to R$ by the formula $\xi_n((u_i)_i) = u_n$. Then ξ_n is a random variable and

$$P_{\xi_n}(A) = P(\xi_n^{-1}(A)) = m \circ h_n(R \times \cdots \times R \times A)$$
$$= m(x_1(R) \ldots x_n(A)) = m_{x_n}(A)$$

hence

$$P_{\xi_n} = m_{x_n}$$

Moreover,

$$\rho_p(\xi_n, \xi_k)^p = \int_{R^N} |\xi_n - \xi_k|^p \, dP = \int_{R^2} |u - v|^p \, dm \circ h(u, v)$$
$$= \rho(x_n, x_k)^p$$

hence $(\xi_n)_n$ is a Cauchy sequence in the space $L^p(R^N, \sigma(\varphi), P)$. Since this space is complete, there exists $\xi \in L^p$ such that

$$\lim_{n\to\infty} \rho_p(\xi_n, \xi) = 0$$

Then there exists a subsequence $(\xi_{n_i})_i$ such that P-a.e. [3]

$$\xi_{n_i} \to \xi$$

Denote $\eta_i = \xi_{n_i}$, $y_i = x_{n_i}$, and put

$$\bar{x}((-\infty, u)) = \bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n\left(\left(-\infty, u - \frac{1}{p}\right\rangle\right)$$

$$\underline{x}((-\infty, u)) = \bigwedge_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} y_n\left(\left(-\infty, u - \frac{1}{p}\right\rangle\right)$$

Let h_{n_j} be the joint observable of observables y_n, y_j, hence $h_{n_j}((-\infty, u) \times (-\infty, v)) = y_n((-\infty, u))y_j((-\infty, v))$. Let h_{k+i} be the joint observable of $y_1, \ldots, y_{k+i}$. Then for $i > j - k$ we have

$$\begin{aligned} &m\left(\bigwedge_{n=k}^{k+i} h_{n_j}((-\infty, u) \times (-\infty, v))\right) \\ &= m\left(\bigwedge_{n=k}^{k+i} h_{k+i}(\{(u_1, \ldots, u_{k+i}); u_n < u, u_j < v\})\right) \\ &\geq m\left(h_{k+i}\left(\bigcap_{n=k}^{k+i} \{(u_1, \ldots, u_{k+i}); u_n < u, u_j < v\}\right)\right) \\ &= P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}((-\infty, u)) \cap \eta_j^{-1}((-\infty, v))\right) \end{aligned}$$

Therefore

$$\begin{aligned} &m(\bar{x}((-\infty, u))y_j((-\infty, v))) \\ &= \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} y_n((-\infty, u))y_j((-\infty, v))\right) \\ &\geq \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}((-\infty, u))\eta_j^{-1}((-\infty, v))\right) \\ &= P(\xi^{-1}((-\infty, u)) \cap \eta_j^{-1}((-\infty, v))) \end{aligned}$$

Similarly,

$$m(x((-\infty, u))y_j((-\infty, v))) \leq P(\xi^{-1}((-\infty, u)) \cap \eta_j^{-1}((-\infty, v)))$$

Since $\bar{x}((-\infty, u)) \leq \underline{x}((-\infty, u))$, we obtain

$$m(\bar{x}((-\infty, u))y_j((-\infty, v))) = m(x((-\infty, u))y_j((-\infty, v)))$$
$$= P(\xi^{-1}((-\infty, u)) \cap \eta_j^{-1}((-\infty, v)))$$

By ref. 7, Theorem 9.8.4, we conclude that there exists an observable y: $\mathcal{B}(R) \to M$ such that

$$m(y((-\infty, u))y_j((-\infty, v))) = P(\eta^{-1}((-\infty, u)) \cap \eta_j^{-1}((-\infty, u)))$$

Moreover,

$$\int_R |t|^p \, dm_y(t) = \int_R |t|^p \, dP_\xi(t) = \int_{R^N} |\xi|^p \, dP < \infty$$

hence $y \in L^p$. Put

$$F(u, v) = P(\xi^{-1}((-\infty, u)) \cap \eta_j^{-1}((-\infty, v)))$$
$$F_n(u, v) = P(\eta_n^{-1}((-\infty, u)) \cap \eta_j^{-1}((-\infty, v)))$$
$$= m(y_n((-\infty, u))y_j((-\infty, v)))$$

Then [3]

$$F(u, v) = \lim_{n\to\infty} F_n(u, v)$$

Further

$$\rho(y, y_j)^p = \int\int_{R^N} |u - v|^p \, dm \circ h(u, v)$$

where h is the joint distribution of y, y_j. It follows that

$$m(h((-\infty, u) \times (-\infty, v))) = m(y((-\infty, u))y_j((-\infty, v)))$$
$$= P(\xi^{-1}((-\infty, u)) \cap \eta_j^{-1}((-\infty, v))) = F(u, v)$$

Therefore

$$\rho(y, y_j)^p = \int\int_{R^2} |u - v|^p \, dm \circ h(u, v)$$
$$= \int\int_{R^2} |u - v|^p \, dF(u, v)$$
$$= \lim_{n\to\infty} \int\int_{R^2} |u - v|^p \, dF_n(u, v)$$
$$= \lim_{n\to\infty} \rho(y_n, y_j)^p$$

$$= \lim_{n\to\infty} \rho_p(\eta_n, \eta_j)^p = \rho_p(\xi, \xi_{n_j})^p$$

We have constructed an observable $y \in L^p$ and a subsequence $(x_{n_j})_j$ such that $x_{n_j} \to y$. Since $(x_n)_n$ is Cauchy, $(x_n)_n$ also converges to y. ■

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